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Note

Regular embeddings of complete bipartite graphs

Roman Nedela[a,1], Martin Škoviera[b,1,*,2], Andrej Zlatoš[c]

[a] Department of Mathematics, Faculty of Science, Matej Bel University, 975 49 Banská Bystrica, Slovakia

[b] Department of Computer Science, Faculty of Mathematics, Physics and Informatics, Comenius University, 842 48 Bratislava, Slovakia

[c] Department of Mathematics, California Institute of Technology, Pasadena, CA 911 25, USA

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Abstract

We prove that for any prime number p the complete bipartite graph $K_{p,p}$ has, up to isomorphism, precisely one regular embedding on an orientable surface—the well-known embedding with faces bounded by hamiltonian cycles.

Keywords: Regular map; Bipartite graph

A cellular embedding $M:K \hookrightarrow S$ of a connected graph K on an orientable surface S is said to be *regular* if the orientation-preserving automorphism group of M acts regularly (or equivalently, transitively) on the darts of K. We use the standard combinatorial representation of an embedding on a surface based on a pair of permutations R and L acting on the darts: R cyclically permutes the darts directed from each vertex v by following the orientation around v while L is the involution which reverses the direction of each dart. In this representation, an embedding is regular if and only if its *monodromy group* $\mathrm{Mon}(M) = \langle R, L\rangle$ acts regularly on the darts.

The problem of classifying regular embeddings of a given graph was apparently first addressed by Coxeter and Moser in their famous book [2, Chapter 8]. Nevertheless, as early as in 1898 Heffter [3] constructed regular embeddings of complete graphs of prime order. The classification was completed much later by James and Jones [4]. So

* Corresponding author.

E-mail address: skoviera@fmph.uniba.sk (M. Škoviera).

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far, complete graphs remain the only major class of graphs for which the complete classification of regular embeddings has been accomplished.

As far as other classes of graphs are concerned, the most challenging appear to be complete bipartite graphs and n-cube graphs. A variety of regular embeddings of these graphs have been constructed by Nedela and Škoviera [5, Section 8]. From among these, the best known is the embedding of the complete bipartite graph $K_{n,n}$ with n faces, each bounded by a hamiltonian cycle (see [1, p. 134]). This embedding can easily be constructed when $K_{n,n}$ is represented as the Cayley graph $C(G,X)$ where $G=\mathbb{Z}_{2n}$ and $X=\{1,3,\dots,2n-1\}$, the vertex-rotations being induced by the cyclic permutation $(1,3,\dots,2n-1)$ of the generators (see [6, Example 7.1]).

The aim of the present note is to prove that for $K_{p,p}$, with p being prime, this is the only regular embedding.

Theorem. *For p a prime, the complete bipartite graph $K_{p,p}$ admits only one regular embedding up to isomorphism—the one described above.*

Proof. Consider an arbitrary regular embedding of $K_{p,p}$. Let R be the rotation and $G=\langle R,L\rangle$ the monodromy group of this embedding. By regularity, the order of G is $2p^2$, the number of darts.

We partition the dart-set of $K_{p,p}$ into two parts, each consisting of darts with origin in the same partite set of vertices. Now, let us take the subgroup $H=\langle R,LRL\rangle$ of G. Clearly, H stabilises the dart partite sets. Moreover, H is transitive on each partite set, so this subgroup *is* the stabiliser. It follows that $|G:H|=2$, $|H|=p^2$, and that G is the semidirect product of H by the cyclic group $\langle L\rangle$ of order 2.

Since H has order p^2, it is abelian. Moreover, H has two cyclic subgroups $\langle R\rangle$ and $\langle LRL\rangle$ of order p with trivial intersection, so H is isomorphic to $\mathbb{Z}_p\times\mathbb{Z}_p$. Hence $G=(\langle R\rangle\times\langle LRL\rangle)\rtimes\langle L\rangle$.

If Q is the rotation of any other regular embedding of $K_{p,p}$, then its monodromy group has the same structure as above with Q in place of R. Therefore, the assignment $Q\mapsto R$ and $L\mapsto L$ establishes an isomorphism of the monodromy groups, proving that Q and R give rise to isomorphic maps. □

As shown in [5], any regular embedding of $K_{n,n}$ transforms into another regular embedding of the same graph if the vertex rotations in one partite set are replaced by their eth powers where $e^2\equiv 1\pmod{n}$. Apart from $e=1$, for $n=p$ the only other possibility is $e=-1$. The resulting embedding is then the *Petrie dual* of the original one. As $K_{p,p}$ has only one regular embedding, this embedding must be self-Petrie.

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